

Non-denseness of factorable matrix functions

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Abstract

It is proved that for certain algebras of continuous functions on compact abelian groups, the set of factorable matrix functions with entries in the algebra is not dense in the group of invertible matrix functions with entries in the algebra, assuming that the dual abelian group contains a subgroup isomorphic to \mathbb{Z}^3 . These algebras include the algebra of all continuous functions and the Wiener algebra. More precisely, it is shown that infinitely many connected components of the group of invertible matrix functions do not contain any factorable matrix functions, again under the same assumption. Moreover, these components actually are disjoint with the subgroup generated by the triangularizable matrix functions.

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1 Introduction

Let G be a (multiplicative) connected compact abelian group and let Γ be its (additive) character group. Recall that Γ consists of continuous homomorphisms of G into the group \mathbb{T} of unimodular complex numbers. Note that Γ is torsion free and discrete in the natural topology of the dual locally compact group, and conversely every torsion free discrete abelian group is the character group of some connected compact abelian group; see, e.g. [32, Section 1.2].

It is well-known [32] that, because G is connected, Γ can be made into a linearly ordered group. So let \preceq be a fixed linear order such that (Γ, \preceq) is an ordered group. Let $\Gamma_+ = \{x \in \Gamma : x \succeq 0\}$, $\Gamma_- = \{x \in \Gamma : x \preceq 0\}$. Standard widely used examples of Γ are \mathbb{Z} (the group of integers), \mathbb{R} (the group of reals with the discrete topology), and $\mathbb{Z}^k, \mathbb{R}^k$ with a lexicographic ordering (where k is positive integer).

If U is a unital ring, we denote by $U^{n \times n}$ the $n \times n$ matrix ring over U , and by $GL(U^{n \times n})$ the group of invertible elements of $U^{n \times n}$. If in addition U is a topological ring (e.g. unital commutative Banach algebra), then we also denote by $GL_0(U^{n \times n})$ the connected component of I_n (the $n \times n$ identity matrix) in $GL(U^{n \times n})$.

Let $C(G)$ be the algebra of (complex valued) continuous functions on G , and let $P(G)$ be the (non-closed) subalgebra of $C(G)$ of all finite linear combinations of functions $\langle j, \cdot \rangle$, $j \in \Gamma$, where $\langle j, g \rangle$ stands for the action of the character $j \in \Gamma$ on the group element $g \in G$ (thus, $\langle j, g \rangle \in \mathbb{T}$). Since Γ is written additively and G multiplicatively, we have

$$\begin{aligned}\langle \alpha + \beta, g \rangle &= \langle \alpha, g \rangle \cdot \langle \beta, g \rangle, & \alpha, \beta \in \Gamma, \quad g \in G, \\ \langle \alpha, gh \rangle &= \langle \alpha, g \rangle \cdot \langle \alpha, h \rangle, & \alpha \in \Gamma, \quad g, h \in G.\end{aligned}$$

In the case $\Gamma = \mathbb{Z}$, $G = \mathbb{T}$ (the unit circle), we will also use the familiar notation $\langle j, e^{it} \rangle = e^{ijt}$, $0 \leq t < 2\pi$, $j \in \mathbb{Z}$. Note that $P(G)$ is dense in $C(G)$ (indeed, by the Pontryagin duality — G is the dual of Γ — Γ separates points in G ; now use the Stone-Weierstrass theorem). For

$$a = \sum_{k=1}^m a_{j_k} \langle j_k, \cdot \rangle \in P(G), \quad j_1, \dots, j_k \in \Gamma \text{ are distinct; } \quad a_{j_k} \neq 0, \quad k = 1, 2, \dots, m,$$

the *Bohr-Fourier spectrum* is defined as the finite set

$$\sigma(a) := \{j_1, \dots, j_k\},$$

and the a_{j_k} 's are the *Bohr-Fourier coefficients* of a . The notions of Bohr-Fourier coefficients and Bohr-Fourier spectrum are extended from functions in $P(G)$ to $C(G)$ by

continuity. The Bohr-Fourier spectrum of $A = [a_{ij}]_{i,j=1}^n \in C(G)^{n \times n}$ is, by definition, the union of the Bohr-Fourier spectra of the a_{ij} 's. Note that the Bohr-Fourier spectra of elements of $C(G)$ are at most countable; a proof for the case $\Gamma = \mathbb{R}$ is found, for example, in [10, Theorem 1.15]; it can be easily extended to general connected compact abelian groups G .

We say that a unital Banach algebra $\mathcal{B} \subseteq C(G)$ is *admissible* if the following properties are satisfied:

- (1) $P(G)$ is dense in \mathcal{B} ;
- (2) \mathcal{B} is inverse closed (i.e. $X \in \mathcal{B} \cap GL(C(G))$ implies $X \in GL(\mathcal{B})$).

It is easy to see that property (2) of admissible algebras extends to matrix functions:

$$\mathcal{B}^{n \times n} \cap GL(C(G)^{n \times n}) = GL(\mathcal{B}^{n \times n}). \quad (1.1)$$

Indeed, $A \in GL(C(G)^{n \times n})$ if and only if $A \in C(G)^{n \times n}$ and $\det A \in GL(C(G))$.

Since $P(G)$ is dense in $C(G)$, it follows that an admissible algebra is dense in $C(G)$. Important examples of admissible algebras are $C(G)$ itself and the *Wiener algebra* $W(G)$ that consists of all functions a on G of the form

$$a(g) = \sum_{j \in \Gamma} a_j \langle j, g \rangle, \quad g \in G, \quad (1.2)$$

where $a_j \in \mathbb{C}$ and $\sum_{j \in \Gamma} |a_j| < \infty$. The norm in $W(G)$ is defined by

$$\|a\|_1 = \sum_{j \in \Gamma} |a_j| < \infty.$$

The inverse closed property of $W(G)$ follows from the Bochner-Philips theorem [2] (a generalization of the classical Wiener's theorem from the case when $G = \mathbb{T}$).

For an admissible algebra \mathcal{B} , we denote by \mathcal{B}_\pm the closed unital subalgebra of \mathcal{B} formed by elements of \mathcal{B} with the Bohr-Fourier spectrum in Γ_\pm (the closedness of \mathcal{B}_\pm follows from the lower semicontinuity of the Bohr-Fourier spectrum: if $a_m \rightarrow a$ in \mathcal{B} as $m \rightarrow \infty$ and $\sigma(a_m) \subseteq K$ for all m , where $K \subseteq \Gamma$ is fixed, then also $\sigma(a) \subseteq K$). Thus, $\mathcal{B}_\pm = \mathcal{B} \cap C(G)_\pm$. Also,

$$GL(C(G)_\pm^{n \times n}) \cap \mathcal{B}^{n \times n} = GL(\mathcal{B}_\pm^{n \times n}). \quad (1.3)$$

Indeed, \supseteq is obvious. For the converse, if $A \in GL(C(G)_+^{n \times n}) \cap \mathcal{B}^{n \times n}$, then $A^{-1} \in C(G)_+^{n \times n} \cap \mathcal{B}^{n \times n} \subseteq \mathcal{B}_+^{n \times n}$, where the inclusion follows from (1.1).

Next, we recall the concept of factorization in the compact abelian group setting, see e.g. [25, 24, 12]. Let \mathcal{B} be an admissible algebra, and let $A \in \mathcal{B}^{n \times n}$. A representation of the form

$$A(g) = A_-(g) (\text{diag}(\langle j_1, g \rangle, \dots, \langle j_n, g \rangle)) A_+(g), \quad g \in G, \quad (1.4)$$

where $A_{\pm}, A_{\pm}^{-1} \in \mathcal{B}_{\pm}^{n \times n}$ and $j_1, \dots, j_n \in \Gamma$, is called a (right - with respect to the order \preceq) \mathcal{B} -factorization of A . Since $C(G)_+ \cap C(G)_-$, and hence also $\mathcal{B}_+ \cap \mathcal{B}_-$, consists of constants only, it follows that the elements j_1, \dots, j_n in (1.4) are uniquely determined by A , up to a permutation. The element $j_1 + \dots + j_n \in \Gamma$ is the *mean motion* of A , denoted $\text{MM}(A)$. The factorization (1.4) is called *canonical* if all j_1, \dots, j_n are zeros. Note that (1.4) yields (by taking determinants) a \mathcal{B} -factorization of $\det(A(g))$; in particular, $\text{MM}(A) = \text{MM}(\det(A))$.

We say that $A \in \mathcal{B}^{n \times n}$ is \mathcal{B} -factorable, resp. *canonically \mathcal{B} -factorable*, if a \mathcal{B} -factorization, resp. canonical \mathcal{B} -factorization, of A exists. Clearly, it is necessary that $A \in GL(\mathcal{B}^{n \times n})$ for A to be \mathcal{B} -factorable.

If $\Gamma = \mathbb{Z}$, then $G = \mathbb{T}$, and $W(\mathbb{T})$ -factorization is the classical Wiener-Hopf factorization on the unit circle. As it happens, in this case the above mentioned necessary invertibility condition is sufficient as well. This result is due to Gohberg-Krein [16], and can also be found in many monographs, e.g. [9, 23]. It is well known that the condition $A \in GL(C(\mathbb{T})^{n \times n})$ is not sufficient for $C(\mathbb{T})$ -factorization even when $n = 1$; an example can be found e.g. in [18].

For $\Gamma = \mathbb{R}$ the dual group G is the Bohr compactification of \mathbb{R} , so that $C(G)$ is nothing but the algebra AP of Bohr almost periodic functions while $W(G)$ is its (non-closed) subalgebra APW of AP functions with absolutely convergent Bohr-Fourier series. Characters $\langle j, \cdot \rangle$ in this setting are simply the exponentials $e_j(x) = \exp(ijx)$. The \mathcal{B} -factorization corresponding to these cases, called AP and APW factorization, respectively, in the matrix setting was first treated in [21, 22]. It was then observed that there exist matrix functions in $GL(APW^{2 \times 2})$ which are not AP -factorable. In other words, the necessary invertibility condition in general is not sufficient. The detailed construction of such matrix functions can be found in [4], while more recent new classes are discussed in [7, 8, 28].

In each of these classes, however, the resulting sets of non APW factorable matrix functions are nowhere dense. This fact has lead many researchers to view the following conjecture as plausible: *For every admissible algebra \mathcal{B} , the set $GLF(\mathcal{B}^{n \times n})$ of \mathcal{B} -factorable elements in $\mathcal{B}^{n \times n}$ is dense in $GL(\mathcal{B}^{n \times n})$.* This is the main issue we are addressing in the present paper.

The conjecture holds for scalar valued functions; this is part of Theorem 7.1 below. (In order not to interrupt the main stream of the paper, the scalar case is relegated to Section 7.)

Our main, and rather surprising, finding is that the conjecture is not true in the matrix case. Namely, the conjecture fails for any Γ that contains a copy of \mathbb{Z}^3 . Moreover, we prove that in this case the minimal closed subgroup of $GL(\mathcal{B}^{n \times n})$ containing the set of \mathcal{B} -factorable matrix functions, is not dense in $GL(\mathcal{B}^{n \times n})$ (Theorem 2.1). Even more, this non-denseness result can be extended to the set of triangularizable matrix functions (Theorem 8.1). Note also that the set $GLF(\mathcal{B}^{n \times n})$ is not necessarily open.

For example, it was shown in [7] that triangular matrix functions

$$\begin{bmatrix} e_\lambda & f \\ 0 & e_{-\lambda} \end{bmatrix}$$

with $f \in APW$ such that $\sigma(f)$ is disjoint with some interval $I \subset (-\lambda, \lambda)$ of length $d \geq \lambda$ are APW factorable if and only if the endpoints of I belong to $\sigma(f)$. Consequently, for $d > \lambda$ such matrix functions lie on the boundary of $GLF(APW^{2 \times 2})$.

Throughout the paper we denote by $\text{diag}(X_1, \dots, X_m)$ the block diagonal matrix with the diagonal blocks X_1, \dots, X_m , in that order.

2 Description of results

Our main result is as follows.

Theorem 2.1 *Let Γ be a torsion free abelian group that contains a subgroup isomorphic to \mathbb{Z}^3 , and let \mathcal{B} be an admissible algebra. Then, for every natural $n \geq 2$ there exist infinitely many pathwise connected components of $GL(\mathcal{B}^{n \times n})$ with the property that each one of these components does not intersect the minimal closed subgroup of $GL(\mathcal{B}^{n \times n})$ containing $GLF(\mathcal{B}^{n \times n})$; in particular, the minimal closed subgroup of $GL(\mathcal{B}^{n \times n})$ containing $GLF(\mathcal{B}^{n \times n})$ is not dense in $GL(\mathcal{B}^{n \times n})$.*

In Theorem 2.1, the closedness of a subgroup is understood relative to $GL(\mathcal{B}^{n \times n})$.

Of course, the non-denseness of \mathcal{B} -factorable matrix functions is its easy consequence.

Note that the result of Theorem 2.1 does not depend on the order \preceq .

The proof of Theorem 2.1 is contained in Sections 3–6. Before passing to their more detailed description, note that in contrast to Theorem 2.1, for the group of rationals and its subgroups, the \mathcal{B} -factorable matrix functions are dense:

Theorem 2.2 *If Γ is isomorphic to a subgroup of the additive group of rational numbers \mathbb{Q} (with the discrete topology), and \mathcal{B} is an admissible algebra, then $GLF(\mathcal{B}^{n \times n})$ is dense in $\mathcal{B}^{n \times n}$.*

Proof. We can approximate any given element of $\mathcal{B}^{n \times n}$ with elements in $P(G)^{n \times n}$ (property (1) of the definition of an admissible algebra). Now use the property that the subgroup of \mathbb{Q} generated by the finite Bohr-Fourier spectrum of any element of $P(G)^{n \times n}$ is isomorphic to \mathbb{Z} . Thus, it suffices to show that, for the case $\Gamma = \mathbb{Z}$, $G = \mathbb{T}$, if

$$a(t) = \sum_{j=j_0}^{j_1} a_j e^{ijt}, \quad 0 \leq t < 2\pi, \quad a_{j_0}, a_{j_0+1}, \dots, a_{j_1} \in \mathbb{C}^{n \times n},$$

then for some integer $j_2 \geq j_1$ the following property holds: for every $\epsilon > 0$ there exist $a'_k \in \mathbb{C}^{n \times n}$, $k = j_0, \dots, j_2$, such that $\|a'_k - a_k\| < \epsilon$ for $k = j_0, \dots, j_2$ (we take $a_{j_1+1} = \dots = a_{j_2} = 0$) and the matrix function $a'(t) := \sum_{j=j_0}^{j_2} a'_j e^{ijt}$ is \mathcal{B} -factorable. We may assume without loss of generality that $j_0 = 0$. Using the Smith form (the diagonal form) of the matrix polynomial

$$\widehat{a}(z) := \sum_{j=0}^{j_1} a_j z^j,$$

we see that for some integer $j_2 \geq j_1$, for every $\epsilon > 0$ there exist $a'_k \in \mathbb{C}^{n \times n}$, $k = 0, 1, \dots, j_2$, such that $\|a'_k - a_k\| < \epsilon$ for $k = j_0, \dots, j_2$ and $\det a'(t) \neq 0$ for all $t \in [0, 2\pi)$. The standard procedure for factorization of rational matrix functions (see e.g. [14, Section XIII.2]) now implies that a' is \mathcal{B} -factorable. \square

For groups Γ that are not subgroups of \mathbb{Q} and at the same time do not contain \mathbb{Z}^3 (example: $\Gamma = \mathbb{Z}^2$), we do not know whether or not $GLF(\mathcal{B}^{n \times n})$ is dense in $GL(\mathcal{B}^{n \times n})$, where \mathcal{B} is an admissible algebra. In this respect note that every pathwise connected component of $GL(C(\mathbb{T}^2)^{n \times n})$ contains $C(\mathbb{T}^2)$ -factorable elements of $C(\mathbb{T}^2)^{n \times n}$, in contrast to Theorem 5.1. This can be easily seen by comparing Proposition 3.4 and Theorem 5.2; indeed, the pathwise connected components of $GL(C(\mathbb{T}^2)^{n \times n})$, as well as those of the set of $C(\mathbb{T}^2)$ -factorable elements of $C(\mathbb{T}^2)^{n \times n}$, are parametrized identically by \mathbb{Z}^2 .

In the next section we develop some preliminary results on pathwise connected components that are perhaps of independent interest. The proof of a preliminary result and some corollaries are given in Section 5. In Section 4 we present an example treating in detail the key case of the three-dimensional torus. In Section 6 we study the minimal subgroup containing all factorable matrix functions and complete the proof of Theorem 2.1. As was mentioned earlier, the scalar case is studied in Section 7. Finally, in Section 8 we extend our main result to non-density of triangularizable matrix functions.

3 Connected components: finite dimensional tori

Consider the following problem:

Problem 3.1 *Describe the pathwise connected components of the group $GL(\mathcal{B}^{n \times n})$, where \mathcal{B} is an admissible algebra.*

Note that the set of pathwise connected components of $GL(\mathcal{B}^{n \times n})$ has a natural group structure: If $\mathcal{C}_1, \mathcal{C}_2$ are two such components, then $\mathcal{C}_1 \cdot \mathcal{C}_2$ is the component defined by the property that $A_1 A_2 \in \mathcal{C}_1 \cdot \mathcal{C}_2$, where $A_1 \in \mathcal{C}_1$, $A_2 \in \mathcal{C}_2$.

To study Problem 3.1, the following proposition will be handy.

Proposition 3.2 *Let \mathcal{B} be an admissible algebra. Then:*

(1) The pathwise connected components of $GL(\mathcal{B}^{n \times n})$ have the form

$$GL(\mathcal{B}^{n \times n}) \cap \mathcal{C}, \quad \mathcal{C} \in C_n(G), \quad (3.1)$$

where $C_n(G)$ stands for the group of pathwise connected components of $GL(C(G)^{n \times n})$.

(2) $GL(\mathcal{B}_{\pm}^{n \times n})$ is pathwise connected.

Proof. Part (1) follows from Arens' theorem [1]; we give an independent self-contained proof. Clearly, if $f_1 : [0, 1] \rightarrow GL(\mathcal{B}^{n \times n})$ is a continuous path, then $f_1(0)$ and $f_1(1)$ belong to the same connected component of $GL(C(G)^{n \times n})$. Thus, any pathwise connected component of $GL(\mathcal{B}^{n \times n})$ is contained in one of the sets (3.1). Next, we prove that if $X, Y \in GL(\mathcal{B}^{n \times n}) \cap \mathcal{C}$ for some $\mathcal{C} \in C_n(G)$, then X and Y are pathwise connected in $GL(\mathcal{B}^{n \times n})$. Let $f_2 : [0, 1] \rightarrow GL(C(G)^{n \times n})$ be a continuous path such that $f_2(0) = X$, $f_2(1) = Y$. We necessarily have $f_2([0, 1]) \subset \mathcal{C}$. Since $GL(C(G)^{n \times n})$ is open, there exists $\varepsilon > 0$ such that

$$Z \in C(G)^{n \times n}, \quad \|Z - f_2(t)\|_{\infty} < \varepsilon \text{ for some } t \in [0, 1] \implies Z \in GL(C(G)^{n \times n}). \quad (3.2)$$

Let m be a sufficiently large integer so that

$$\|f_2((j+1)/m) - f_2(j/m)\|_{\infty} < \varepsilon, \quad j = 0, 1, \dots, m-1. \quad (3.3)$$

Let

$$\kappa := \max_{j=0,1,\dots,m-1} \{\|f_2((j+1)/m) - f_2(j/m)\|_{\infty}\} < \varepsilon, \quad (3.4)$$

and select $\lambda > 0$ such that $\kappa + \lambda < \varepsilon$. Since \mathcal{B} is admissible, we can select $F_{j/m} \in \mathcal{B}^{n \times n}$, $j = 0, 1, \dots, m$ such that $F_0 = X$, $F_1 = Y$, and $\|F_{j/m} - f_2(j/m)\|_{\infty} < \lambda$ for $j = 1, 2, \dots, m-1$. Then (3.3), (3.4) guarantee that

$$[F_{j/m}, F_{(j+1)/m}] := \{tF_{j/m} + (1-t)F_{(j+1)/m} : 0 \leq t \leq 1\}$$

is contained in

$$\{Z \in C(G) : \|Z - f_2((j+1)/m)\|_{\infty} < \varepsilon\} \cap \{Z \in C(G) : \|Z - f_2(j/m)\|_{\infty} < \varepsilon\},$$

for $j = 0, 1, \dots, m-1$. By (3.2), the interval $[F_{j/m}, F_{(j+1)/m}]$ is contained in $GL(C(G)^{n \times n})$, and since \mathcal{B} is admissible, we actually have $[F_{j/m}, F_{(j+1)/m}] \subset GL(\mathcal{B}^{n \times n})$. Now clearly the union $\cup_{j=0}^{m-1} [F_{j/m}, F_{(j+1)/m}]$ represents a continuous path in $GL(\mathcal{B}^{n \times n})$ connecting X and Y .

Finally, observe that $GL(\mathcal{B}^{n \times n})$ is dense in $GL(C(G)^{n \times n})$ (as easily follows from the defining properties of admissible algebras), and therefore $GL(\mathcal{B}^{n \times n}) \cap \mathcal{C} \neq \emptyset$ for every $\mathcal{C} \in C_n(G)$.

Part (2). As in Part (1), using (1.3) we show that the pathwise connected components of $GL(\mathcal{B}_{\pm}^{n \times n})$ are intersections of $GL(\mathcal{B}_{\pm}^{n \times n})$ with the pathwise connected components of $GL(C(G)_{\pm}^{n \times n})$. But the latter is pathwise connected [5, Theorem 6.1], and so we are done. \square

In view of Proposition 3.2, we may assume $\mathcal{B} = C(G)$ in Problem 3.1. In the classical case $\Gamma = \mathbb{Z}$ the answer is known: The group $GL(C(G)^{n \times n})$ has infinitely many pathwise connected components each of which is characterized by a fixed integer – the winding number of the determinant (see, e.g., Bojarski's appendix to [26]). In general, $C_n := GL(C(G)^{n \times n})/GL_0(C(G)^{n \times n})$ can be identified with the group $[G; GL(\mathbb{C}^{n \times n})]$ of homotopy classes of continuous maps $G \rightarrow GL(\mathbb{C}^{n \times n})$.

In this section, we give a solution of Problem 3.1 for the case when G is a finite dimensional torus. The solution for an infinite dimensional G is obtained from here using the fact that G is the inverse limit of a family $\{G_\alpha\}_{\alpha \in \Lambda}$ of finite dimensional tori and hence the set of connected components of $GL(C(G))$ in this case is the direct limit of sets of connected components of $GL(C(G_\alpha))$ under the maps transposed to the corresponding maps of the inverse limit. The solution for G being a finite dimensional torus is based on the classical results of Fox [13] concerning torus homotopy groups and their connections with certain homotopy groups of the unitary group $U_n \subset GL(\mathbb{C}^{n \times n})$.

For the readers' convenience, we recall basic definitions and some results from [13]. We identify the k -dimensional torus \mathbb{T}^k with $\mathbb{R}^k/\mathbb{Z}^k$. Let $E = [0, 1]$ and let E^k stand for the unit k -dimensional cube. Let X be a topological space. We say that a continuous function $f : E^k \rightarrow X$ has *periodic boundary conditions* if for each $i = 1, 2, \dots, k$,

$$f(\dots, x_{i-1}, 0, x_{i+1}, \dots) = f(\dots, x_{i-1}, 1, x_{i+1}, \dots) \quad \forall \quad 0 \leq x_j \leq 1, \quad (j \neq i).$$

Clearly, there is a one-to-one correspondence between such functions f and continuous functions $F : \mathbb{T}^k \rightarrow X$.

Let $o \in X$ be a fixed point. We denote by $\mathcal{T}^k(X, o)$ the set of all continuous functions $f : E^k \rightarrow X$ with periodic boundary conditions which satisfy in addition the condition

$$f(0, x_2, \dots, x_k) = f(1, x_2, \dots, x_k) = o.$$

By $\tau_k(X, o)$ we denote the set of all homotopy classes of functions $f \in \mathcal{T}^k(X, o)$. Then the *torus homotopy group* is the set $\tau_k(X, o)$ together with the group binary operation “+” induced by the operation of multiplication

$$(f_1 \times f_2)(x) = \begin{cases} f_1(2x_1, x_2, \dots, x_k) & \text{if } 0 \leq x_1 \leq 1/2, \\ f_2(2x_1 - 1, x_2, \dots, x_k) & \text{if } 1/2 \leq x_1 \leq 1, \end{cases} \quad (f_1, f_2 \in \mathcal{T}^k(X, o)),$$

with the inverse operation being

$$f^{[-1]}(x) = f(1 - x_1, x_2, \dots, x_k), \quad f \in \mathcal{T}^k(X, o).$$

If X is pathwise connected, then $\tau_k(X, o)$ does not depend on the particular choice of o (up to a group isomorphism).

Proposition 3.3 [13] *If X is a topological group with the unit element e , then $(\tau_k(X, e), +)$ is an abelian group.*

In the sequel we will work with the torus homotopy groups of U_n and use the notation $\tau_k(U_n)$ for $(\tau_k(U_n, I), +)$. One can introduce another multiplication “ \cdot ” on $\tau_k(U_n)$ induced from the matrix multiplication in U_n . Note that $\tau_1(U_n)$ is the fundamental group of U_n , and so $\tau_1(U_n) \cong \mathbb{Z}$. In this case one also has $\tau_1(U_n) \cong (\tau_1(U_n), \cdot)$ but in general for $k, n \geq 2$ these two groups are not isomorphic. In what follows we will use the notation $\dot{\tau}_k(U_n) := (\tau_k(U_n), \cdot)$. Also, we will use the well-known semidirect products of groups. Briefly, a group X is a semidirect product of its normal subgroup N and a subgroup Y , notation: $X = Y \ltimes N$, if $X = NY = YN$ and the intersection of N and Y is trivial; see e.g. [11] for more information.

Proposition 3.4 *Assume $\Gamma = \mathbb{Z}^k$, i.e. $G = \mathbb{T}^k$, and let \mathcal{B} be an admissible algebra. Then the group of pathwise connected components of $GL(\mathcal{B}^{n \times n})$, $n \geq 1$, equipped with the multiplication induced by the product of matrices in $GL_n(\mathbb{C})$, is isomorphic to $\dot{\tau}_k(U_n) \ltimes (\dot{\tau}_{k-1}(U_n) \ltimes \cdots \ltimes (\dot{\tau}_2(U_n) \ltimes \dot{\tau}_1(U_n)) \cdots)$ with naturally defined multiplications in the semidirect products. If $k \leq 2n - 1$, then the direct product of groups $\tau_k(U_n) \times \tau_{k-1}(U_n) \times \cdots \times \tau_1(U_n)$ is isomorphic to $\mathbb{Z}^{2^{k-1}}$.*

Proof. In view of Proposition 3.2, we may assume $\mathcal{B} = C(G)$. Note that U_n is a strong deformation retract [19] of $GL(\mathbb{C}^{n \times n})$; indeed, the map

$$(X, t) \mapsto X(X^*X)^{-1/2}(t(X^*X)^{1/2} + (1-t)I), \quad X \in GL(\mathbb{C}^{n \times n}), \quad 0 \leq t \leq 1,$$

provides such a retracting deformation. Therefore, the sets of pathwise connected components of $GL(C(\mathbb{T}^k)^{n \times n})$ and of $U(C(\mathbb{T}^k)^{n \times n})$ ($:=$ the set of continuous functions on \mathbb{T}^k with values in U_n) coincide.

By x_1, \dots, x_k we denote the local coordinates on \mathbb{T}^k induced from the coordinates on \mathbb{R}^k by means of the quotient map $\mathbb{R}^k \rightarrow \mathbb{R}^k/\mathbb{Z}^k = \mathbb{T}^k$. Consider the subtorus \mathbb{T}^{k-1} of codimension 1 in \mathbb{T}^k defined by $\{(x_1, x_2, \dots, x_k) \in \mathbb{T}^k : x_k = 0\}$. By $p_{k-1} : \mathbb{T}^k \rightarrow \mathbb{T}^{k-1}$ we denote the projection sending a point with coordinates (x_1, \dots, x_k) to the point with coordinates $(x_1, \dots, x_{k-1}, 0)$. Each function $X \in U(C(\mathbb{T}^k)^{n \times n})$ can be uniquely written as

$$X = X_1 \cdot X_2, \quad \text{where} \quad X_2 := (X|_{\mathbb{T}^{k-1}}) \circ p_{k-1}, \quad X_1 := X \cdot X_2^{-1}.$$

The matrix X_1 belongs to the Banach group $U_I(C_1(\mathbb{T}^k)^{n \times n})$ of continuous functions on \mathbb{T}^k with values in U_n identically equal I_n on \mathbb{T}^{k-1} . Here $C_1(\mathbb{T}^k)$ is the Banach algebra of (complex) continuous functions on $G := \mathbb{T}^k$ which are constant on \mathbb{T}^{k-1} . (Recall that a Banach group is a Lie group that is modelled locally by open balls in a Banach space.) Clearly, $U_I(C_1(\mathbb{T}^k)^{n \times n})$ is a normal subgroup of $U(C(\mathbb{T}^k)^{n \times n})$ with the quotient group $U(C(\mathbb{T}^{k-1})^{n \times n})$. Moreover the above decomposition shows that the exact sequence of Banach groups

$$1 \longrightarrow U_I(C_1(\mathbb{T}^k)^{n \times n}) \longrightarrow U(C(\mathbb{T}^k)^{n \times n}) \longrightarrow U(C(\mathbb{T}^{k-1})^{n \times n}) \longrightarrow 1 \quad (3.5)$$

splits, i.e. $U(C(\mathbb{T}^k)^{n \times n})$ is a semidirect product of subgroups $U_I(C_1(\mathbb{T}^k)^{n \times n})$ and

$$(p_{k-1})^*U(C(\mathbb{T}^{k-1})^{n \times n}) \cong U(C(\mathbb{T}^{k-1})^{n \times n}).$$

Here, $p_{k-1}^* : U(C(\mathbb{T}^{k-1})^{n \times n}) \rightarrow U(C(\mathbb{T}^k)^{n \times n})$ is defined by

$$(p_{k-1}^*(f))(x) = f(p_{k-1}(x)), \quad f \in U(C(\mathbb{T}^{k-1})^{n \times n}), \quad x \in \mathbb{T}^k.$$

Recall that the set of pathwise connected components of a Banach group Q can be naturally identified with the (discrete) group defined as the quotient of Q by the pathwise connected component containing the identity (which is a normal subgroup of Q). In the case of $U(C(\mathbb{T}^k)^{n \times n})$, the group of pathwise connected components $C_{k,n}$ is a discrete countable group (because the former group is separable). Denote by $\tilde{C}_{k,n}$ the normal subgroup of $C_{k,n}$ isomorphic to the group of pathwise connected components of the Banach group $U_I(C_1(\mathbb{T}^k)^{n \times n})$. Note that we have a split exact sequence

$$1 \longrightarrow \tilde{C}_{k,n} \longrightarrow C_{k,n} \longrightarrow C_{k-1,n} \longrightarrow 1.$$

This follows from (3.5) and the fact that the connected component of $U(C(\mathbb{T}^k)^{n \times n})$ containing the unit matrix is the semidirect product (under the multiplication given by product of matrices) of the connected components containing the unit matrix of groups $U(C(\mathbb{T}^{k-1})^{n \times n})$ and $U_I(C_1(\mathbb{T}^k)^{n \times n})$. Moreover, this connected component of $U_I(C_1(\mathbb{T}^k)^{n \times n})$ is a normal subgroup of $U(C(\mathbb{T}^k)^{n \times n})$. Thus, $C_{k,n}$ is isomorphic to a semidirect product of discrete subgroups $\tilde{C}_{k,n}$ and $C_{k-1,n}$, where $\tilde{C}_{k,n}$ is a normal subgroup of $C_{k,n}$ isomorphic to the group of pathwise connected components of the Banach group $U_I(C_1(\mathbb{T}^k)^{n \times n})$, and $C_{k-1,n}$ is isomorphic to the group of pathwise connected components of $U(C(\mathbb{T}^{k-1})^{n \times n})$. We parametrize elements of $C_{k,n}$ by points of $\tilde{C}_{k,n} \times C_{k-1,n}$. Namely, each element of $C_{k,n}$ can be uniquely presented as a product of elements of $\tilde{C}_{k,n}$ and $C_{k-1,n}$ (taken in this order) and so $C_{k,n}$ can be recovered from $\tilde{C}_{k,n} \times C_{k-1,n}$.

Next, the set of pathwise connected components of $U(C_1(\mathbb{T}^k)^{n \times n})$ is naturally identified with the set of homotopy classes of continuous maps of the maximal ideal space M_1 of the Banach algebra $C_1(\mathbb{T}^k)$ into U_n . By the definition M_1 is homeomorphic to the compact Hausdorff space obtained from \mathbb{T}^k by contracting the subtorus \mathbb{T}^{k-1} into a point. Therefore the set of pathwise connected components of $U_I(C_1(\mathbb{T}^k)^{n \times n})$ is identified with the homotopy classes of continuous maps of \mathbb{T}^k into U_n which map \mathbb{T}^{k-1} into $I_n \in U_n$. Then, by the definition of the multiplication in $\tilde{C}_{k,n}$ we have $\tilde{C}_{k,n} \cong \dot{\tau}_k(U_n)$.

Applying the same arguments to $U(C(\mathbb{T}^{k-1})^{n \times n}), \dots, U(C(\mathbb{T}^1)^{n \times n})$ we obtain the first statement of the proposition. In particular, each pathwise connected component of $GL(C(\mathbb{T}^k)^{n \times n})$ has the form $X_k \cdots X_1 \cdot GL_0(C(\mathbb{T}^k)^{n \times n})$, where X_r represents an element of $\mathcal{T}^r(U_n, I)$, $1 \leq r \leq k$, and $GL_0(C(\mathbb{T}^k)^{n \times n})$ is the pathwise connected component containing I_n .

Further, it is known that $\tau_r(U_n)$, $r \geq 2$, is defined from the split exact sequence of abelian groups, see [13, Statement (9.3)],

$$0 \longrightarrow \prod_{i=2}^r (\pi_i(U_n))^{\alpha_i} \longrightarrow \tau_r(U_n) \longrightarrow \tau_{r-1}(U_n) \longrightarrow 0, \quad (3.6)$$

where $\alpha_i := \binom{r-2}{i-2}$, $\pi_i(U_n)$ is the i -dimensional homotopy group of U_n and $(\pi_i(U_n))^{\alpha_i}$ is the α_i -fold direct product $\pi_i(U_n) \times \cdots \times \pi_i(U_n)$.

To compute $\pi_i(U_n)$ one can use e.g. the Bott periodicity theorem [3] which, in particular, implies that

$$\pi_i(U_n) = \pi_{i+2}(U_{n+1}) \quad \text{for} \quad 0 \leq i \leq 2n-1. \quad (3.7)$$

For instance, using the fact that $\pi_0(U_n) = 0$ (because U_n is pathwise connected) and $\pi_2(U_n) = 0$ (see e.g. [20, Section 8.12]), we obtain from the latter periodicity that the set of pathwise connected components of $GL(C(\mathbb{T}^1)^{n \times n})$ can be naturally identified with $\tau_1(U_n) \cong \mathbb{Z}$ (as remarked above), the set of pathwise connected components of $GL(C(\mathbb{T}^2)^{n \times n})$ can be naturally identified with $\tau_2(U_n) \times \tau_1(U_n) \cong \mathbb{Z} \times \mathbb{Z}$ [33], for $n \geq 2$ the set of pathwise connected components of $GL(C(\mathbb{T}^3)^{n \times n})$ can be naturally identified with $\tau_3(U_n) \times \tau_2(U_n) \times \tau_1(U_n) \cong (\mathbb{Z} \times \mathbb{Z}) \times \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^4$, whereas if $n = 1$ it is identified with \mathbb{Z}^3 , etc.

In particular, if $k \leq 2n-1$, then the set of pathwise connected components of $GL(C(\mathbb{T}^k)^{n \times n})$ is parametrized by $\tau_k(U_n) \times \cdots \times \tau_1(U_n) \cong \mathbb{Z}^{2^{k-1}}$. This easily follows by induction from (3.6), (3.7) using the fact that the sum of all odd binomial coefficients among $\binom{r-2}{i-2}$ is equal to 2^{r-3} , $r \geq 3$. \square

4 Example

In the case of \mathbb{T}^3 we give an explicit description of the identification in Proposition 3.4. This will be later used in the proof of Theorem 5.1.

Example 4.1 As in the proof of Proposition 3.4, we denote by $U_I(C_1(\mathbb{T}^3)^{n \times n})$ the Banach group of continuous functions on \mathbb{T}^3 with values in U_n assuming the value I_n on \mathbb{T}^2 . Let $X \in U_I(C_1(\mathbb{T}^3)^{n \times n})$, $n \geq 2$, be a continuous $n \times n$ unitary matrix function sending $T_3 := \mathbb{T}^2 = \{(x_1, x_2, x_3) \in \mathbb{T}^3 : x_3 = 0\} \subset \mathbb{T}^3$ into $I_n \in U_n$. We will consider also subtori

$$T_2 := \{(x_1, x_2, x_3) \in \mathbb{T}^3 : x_2 = 0\} \quad \text{and} \quad T_1 := \{(x_1, x_2, x_3) \in \mathbb{T}^3 : x_1 = 0\}$$

of \mathbb{T}^3 with projections

$$t_2 : \mathbb{T}^3 \rightarrow T_2, \quad (x_1, x_2, x_3) \mapsto (x_1, 0, x_3) \quad \text{and} \quad t_1 : \mathbb{T}^3 \rightarrow T_1, \quad (x_1, x_2, x_3) \mapsto (0, x_2, x_3),$$

respectively. We write

$$X = X_1 \cdot X_2 \cdot X_3, \quad X_j \in U_I(C_1(\mathbb{T}^3)^{n \times n}), \quad j = 1, 2, 3,$$

where

$$\begin{aligned} X_3(x_1, x_2, x_3) &= X(x_1, 0, x_3), \\ X_2(x_1, x_2, x_3) &= X(0, x_2, x_3)X(0, 0, x_3)^{-1}, \\ X_1(x_1, x_2, x_3) &= X(x_1, x_2, x_3)X(x_1, 0, x_3)^{-1}X(0, 0, x_3)X(0, x_2, x_3)^{-1}. \end{aligned}$$

Considering X_1, X_2, X_3 as U_n -valued periodic functions on \mathbb{R}^3 we obtain that X_1 sends the boundary ∂E^3 of the three dimensional closed unit cube E^3 into I_n ; X_2 is the pullback by t_1 of some periodic unitary matrix function \tilde{X}_2 defined on the plane $R_1 := \{(0, x_2, x_3) \in \mathbb{R}^3\}$ such that

$$\tilde{X}_2(0, x_2, 0) = \tilde{X}_2(0, x_2, 1) = \tilde{X}_2(0, 0, x_3) = \tilde{X}_2(0, 1, x_3) = I_n,$$

i.e.,

$$X_2(x) = \tilde{X}_2(t_1(x)), \quad x \in \mathbb{T}^3;$$

and X_3 is the pullback by t_2 of some periodic unitary matrix function \tilde{X}_3 defined on the plane $R_2 := \{(x_1, 0, x_3) \in \mathbb{R}^3\}$ such that $\tilde{X}_3(x_1, 0, 0) = \tilde{X}_3(x_1, 0, 1) = I_n$.

Let \dot{E}^3 be the interior of the three dimensional closed cube E^3 . The one-point compactification of \dot{E}^3 is homeomorphic to the unit three dimensional sphere $\mathbb{S}^3 \subset \mathbb{R}^4$. Moreover, there exists a continuous surjection $E^3 \rightarrow \mathbb{S}^3$ which maps the boundary of ∂E^3 of the cube into the point $P = (1, 0, 0, 0)$ and $E^3 \setminus \partial E^3$ bijectively onto $\mathbb{S}^3 \setminus P$. This map can be extended by periodicity to \mathbb{R}^3 and this extension determines a continuous surjective map $\psi : \mathbb{T}^3 \rightarrow \mathbb{S}^3$ such that the induced map of the Čech cohomology groups

$$\psi^* : H^3(\mathbb{S}^3, \mathbb{Z}) \cong \mathbb{Z} \rightarrow H^3(\mathbb{T}^3, \mathbb{Z}) \cong \mathbb{Z}$$

is an isomorphism. (That ψ^* is an isomorphism follows from ψ being a bi-continuous bijection between open dense subsets $\dot{E}^3 \subset \mathbb{T}^3$ and $\mathbb{S}^3 \setminus P \subset \mathbb{S}^3$.) By the definition of the function X_1 , there exists a continuous function X'_1 on \mathbb{S}^3 with values in U_n whose pullback by ψ coincides with X_1 and such that $X'_1(P) = I_n$. In particular, X'_1 determines an element of the homotopy group $\pi_3(U_n)$. Similarly \tilde{X}_2 maps the boundary of the square $E^3 \cap R_1$ into I_n and therefore is the pullback of a function of the two-dimensional unit sphere \mathbb{S}^2 into U_n and so determines an element of $\pi_2(U_n)$. But the latter group is trivial and therefore \tilde{X}_2 considered as a map of the two-dimensional torus into U_n is homotopic to a constant map implying that X_2 belongs to the connected component of $U_I(C_1(\mathbb{T}^3)^{n \times n})$ containing the constant map.

Further, analogously to X the function \tilde{X}_3 can be factorized as

$$\tilde{X}_3 = \tilde{X}_{31} \cdot \tilde{X}_{32},$$

where \tilde{X}_{31} maps the boundary of the square $E^3 \cap R_2$ into I_n and therefore as above (since $\pi_2(U_n) = 0$) $\tilde{X}_{31} \circ t_2$ belongs to the connected component of $U_I(C_1(\mathbb{T}^3)^{n \times n})$ containing the constant map, and \tilde{X}_{32} is the pullback by t_1 of a continuous periodic unitary function \tilde{X}'_{32} defined on the line $\{(0, 0, x_3) \in \mathbb{R}^3\}$. This function by definition determines an element of $\pi_1(U_n) \cong \mathbb{Z}$. Moreover, each such \tilde{X}'_{32} is homotopic to a function $\text{diag}(1, \dots, 1, e^{2\pi i k x_3})$ for some $k \in \mathbb{Z}$. Combining the above facts we obtain that

$$X = X' \cdot \text{diag}(1, \dots, 1, e^{2\pi i k x_3}) \cdot X'', \quad (4.1)$$

where X' is the pullback by ψ of a continuous map $\tilde{X}' : \mathbb{S}^3 \rightarrow U_n$ representing an element of $\pi_3(U_n)$ and X'' belongs to the connected component of $U_I(C_1(\mathbb{T}^3)^{n \times n})$ containing the constant map. Note that k and the element of $\pi_3(U_n)$ in (4.1) are unique, for a given X , but the factorization (4.1) is generally not unique (this fact will not be used in the sequel).

Let us describe some canonical matrix functions representing the first terms of the above factorization.

Consider the group action of U_n on \mathbb{S}^{2n-1} defined by the map $p : U_n \times \mathbb{S}^{2n-1} \rightarrow \mathbb{S}^{2n-1}$,

$$p(X, (s_1, \dots, s_{2n})) = X \begin{bmatrix} s_1 + is_2 \\ s_3 + is_4 \\ \vdots \\ s_{2n-1} + is_{2n} \end{bmatrix},$$

where

$$X \in U_n, \quad s_1, \dots, s_{2n} \in \mathbb{R}, \quad s_1^2 + \dots + s_{2n}^2 = 1.$$

For $e_1 := (1, 0, \dots, 0) \in \mathbb{S}^{2n-1}$ the subgroup of U_n fixing e_1 is $1 \times U_{n-1} \cong U_{n-1}$. The group $1 \times U_{n-1}$ acts on U_n by left multiplications and the set of equivalence classes $U_n / (1 \times U_{n-1})$ under this action is a homogeneous space diffeomorphic to \mathbb{S}^{2n-1} . The latter diffeomorphism is defined by identifying the first column of each matrix from U_n with an element of \mathbb{S}^{2n-1} . Now the composition π of maps $U_n \rightarrow U_n / (1 \times U_{n-1}) \rightarrow \mathbb{S}^{2n-1}$ defines U_n as a principal fibre bundle over \mathbb{S}^{2n-1} with fibre $U_{n-1} \cong 1 \times U_{n-1} \subset U_n$. In particular, over any proper open subset S of \mathbb{S}^{2n-1} there exists a smooth right inverse to the map π (because $\mathbb{S}^{2n-1} \setminus \{s\}$ is contractible for each $s \in \mathbb{S}^{2n-1}$). This implies that $\pi^{-1}(S)$ is diffeomorphic to $S \times U_{n-1}$. Using this construction one can easily show (comparing dimensions of \mathbb{S}^3 and of \mathbb{S}^{2n-1}) that each continuous map $\mathbb{S}^3 \rightarrow U_n$ is homotopic to a continuous map $\mathbb{S}^3 \rightarrow U_2$, where U_2 is identified with the subgroup $1 \times 1 \times \dots \times 1 \times U_2 \subset U_n$.

Moreover, if two continuous maps $f_0, f_1 : \mathbb{S}^3 \rightarrow 1 \times U_{n-1}$ ($n \geq 3$) are homotopic to each other via a homotopy $f_t : \mathbb{S}^3 \rightarrow U_n$, $0 \leq t \leq 1$, then they are also homotopic to each other via a homotopy $\tilde{f}_t : \mathbb{S}^3 \rightarrow 1 \times U_{n-1}$. Indeed, re-write f_t as a map $f : [0, 1] \times \mathbb{S}^3 \rightarrow U_n$, where

$$f_0(\mathbb{S}^3) = f(0, \mathbb{S}^3) \subseteq 1 \times U_{n-1}, \quad f_1(\mathbb{S}^3) = f(1, \mathbb{S}^3) \subseteq 1 \times U_{n-1}.$$

Using a dimension argument ($n \geq 3$) we may assume that $\pi \circ f : [0, 1] \times \mathbb{S}^3 \rightarrow \mathbb{S}^{2n-1}$ is not surjective; here π is defined in the preceding paragraph. Using the principal fiber bundle construction as described there, we see that there is a homotopy $F : [0, 1] \times [0, 1] \times \mathbb{S}^3$ connecting f to some $\hat{f} : [0, 1] \times \mathbb{S}^3 \rightarrow 1 \times U_{n-1}$, i.e. $F(0, \cdot, \cdot) = f$, $F(1, \cdot, \cdot) = \hat{f}$. Moreover, from the fibre bundle argument it follows that one can make sure that if $f(t, z) \in 1 \times U_{n-1}$, then $F(s, t, z) = f(t, z)$ for all $s \in [0, 1]$. Now \hat{f} is the desired homotopy between f_0 and f_1 .

It follows that the natural map $\tau : \pi_3(U_{n-1}) \rightarrow \pi_3(U_n)$ is an isomorphism, and by induction we see that $\pi_3(U_2)$ is naturally isomorphic to $\pi_3(U_n)$. In turn, U_2 is diffeomorphic to $\mathbb{S}^1 \ltimes \mathbb{S}^3$ (U_2 is a semidirect product of U_1 and of the 2×2 special unitary group $SU_2 \cong \mathbb{S}^3$). Since $\pi_3(U_2) = \pi_3(\mathbb{S}^1 \ltimes \mathbb{S}^3)$ is isomorphic to $\pi_3(SU_2) = \pi_3(\mathbb{S}^3)$ (because $\pi_3(\mathbb{S}^1)$ is trivial), we obtain therefore that each continuous map $\mathbb{S}^3 \rightarrow U_n$ is homotopic to a continuous map $\mathbb{S}^3 \rightarrow \mathbb{S}^3$ under the identification $\mathbb{S}^3 = SU_2$.

Next, consider the map $\mathbb{R}^3 \rightarrow \mathbb{R}^3$, $(x_1, x_2, x_3) \mapsto (kx_1, x_2, x_3)$, $k \in \mathbb{Z}$. For $k \neq 0$, it determines a finite map $d_k : \mathbb{T}^3 \rightarrow \mathbb{T}^3$; the number of preimages of each point under d_k is k and d_k preserves orientation for $k > 0$ and reverses it for $k < 0$. For $k = 0$ it determines a projection $d_0 : \mathbb{T}^3 \rightarrow T_1$, where $T_1 := \{x = (0, x_2, x_3) \in \mathbb{T}^3\}$ is a two-dimensional subtorus. We set

$$\psi_k := \psi \circ d_k.$$

By definitions of ψ and d_k there exists a continuous map $e_k : \mathbb{S}^3 \rightarrow \mathbb{S}^3$ such that $\psi_k = e_k \circ \psi$, for $k \neq 0$ preimage of each point of $\mathbb{S}^3 \setminus \{(1, 0, 0, 0)\}$ under e_k consists of k distinct points, and preimage of $(1, 0, 0, 0)$ has dimension ≤ 2 . Also, ψ_0 has two-dimensional image. By the definition e_k determines a map in cohomology $e_k^* : H^3(\mathbb{S}^3, \mathbb{Z}) \cong \mathbb{Z} \rightarrow H^3(\mathbb{S}^3, \mathbb{Z}) \cong \mathbb{Z}$ such that $e_k^*(m) := km$, $m \in \mathbb{Z}$. Assume now that \tilde{X}' as above represents a nonzero element $s \in \pi_3(U_n) = \pi_3(SU_2) (= \pi_3(\mathbb{S}^3))$. If $k (\neq 0)$ is the degree of s , then by the Hopf theorem (as stated for example in [19, Section II.8]), \tilde{X}' is homotopy equivalent to e_k . From here identifying \mathbb{S}^3 with SU_2 by sending $x = (x_1, x_2, x_3, x_4) \in \mathbb{S}^3$ into the matrix

$$S(x) := \begin{bmatrix} x_1 + ix_2 & -(x_3 - ix_4) \\ x_3 + ix_4 & x_1 - ix_2 \end{bmatrix} \in SU_2$$

we obtain that each $X \in U_I(C_1(\mathbb{T}^3)^{n \times n})$, $n \geq 2$, admits a factorization

$$X = \text{diag}(1, \dots, 1, (S \circ \psi_m)) \cdot \text{diag}(1, \dots, 1, e^{2\pi i p x_3}) \cdot Y$$

where $m, p \in \mathbb{Z}$ and Y belongs to the connected component of $U_I(C_1(\mathbb{T}^3)^{n \times n})$ containing the constant map. The numbers m, p here are uniquely defined but the above factorization is not unique.

5 Connected components without factorable elements

In this section we prove the following result which is a particular case of, and at the same time a stepping stone to, the proof of Theorem 2.1.

Theorem 5.1 *Let Γ be a torsion free abelian group that contains a subgroup isomorphic to \mathbb{Z}^3 . Then for every natural $n \geq 2$, there exist infinitely many pathwise connected components of $GL(C(G)^{n \times n})$ with the property that each one of these components does not contain any $C(G)$ -factorable element of $C(G)^{n \times n}$.*

The following result obtained in [5] will be used in the proof:

Theorem 5.2 *The pathwise connected components of $GLF(C(G)^{n \times n})$ are precisely the sets*

$$\{X \in GLF(C(G)^{n \times n}) : \text{MM}(X) = \gamma\} \quad (5.1)$$

parametrized by $\gamma \in \Gamma$.

In fact each of the sets (5.1) contains the element $\text{diag}(1, \dots, 1, \langle \gamma, \cdot \rangle)$. In other words, each $C(G)$ -factorable matrix function can be path-connected to an element in the above form.

Fix $\tilde{\Gamma} \subset \Gamma$, a subgroup isomorphic to \mathbb{Z}^3 and its generators j_1, j_2, j_3 , and define a continuous surjective group homomorphism $\tilde{p} : G \rightarrow \mathbb{T}^3$ by the formula:

$$\tilde{p}(g) := (\langle j_1, g \rangle, \langle j_2, g \rangle, \langle j_3, g \rangle) \quad g \in G.$$

Let Γ_0 be a finitely generated subgroup containing $\tilde{\Gamma}$. Consider the family \mathcal{F} of all finitely generated subgroups $\Gamma_\alpha \subset \Gamma$, $\alpha \in \Lambda$, containing Γ_0 , where Λ is a suitable index set with the partial order defined by the property that $\alpha \leq \beta$ if and only if $\Gamma_\alpha \subseteq \Gamma_\beta$; the infimal element of Λ is denoted 0 in agreement with our notation Γ_0 . We equip \mathcal{F} with the structure of the direct limiting system using natural inclusions

$$i_{\alpha\beta} : \Gamma_\alpha \hookrightarrow \Gamma_\beta, \quad \alpha \leq \beta,$$

so that the limit of the dual to \mathcal{F} inverse limiting system $\hat{\mathcal{F}}$ of groups $\hat{\Gamma}_\alpha$ which are Pontryagin duals to Γ_α , $\alpha \in \Lambda$, is homeomorphic to G . Since each Γ_α is isomorphic to some $\mathbb{Z}^{k(\alpha)}$, $k(\alpha) \in \mathbb{N}$, the group $\hat{\Gamma}_\alpha$ is isomorphic to the standard $k(\alpha)$ -dimensional torus $\mathbb{T}^{k(\alpha)}$. Thus, G is isomorphic to the inverse limit of tori $\mathbb{T}^{k(\alpha)}$ under some continuous epimorphisms ($:=$ surjective group homomorphisms)

$$p_\alpha^\beta : \mathbb{T}^{k(\beta)} \rightarrow \mathbb{T}^{k(\alpha)}, \quad k(\beta) \geq k(\alpha), \quad \alpha \leq \beta.$$

By $p_\alpha : G \rightarrow \mathbb{T}^{k(\alpha)}$ we denote the corresponding limit epimorphisms.

One of the ways to describe the above construction explicitly is as follows: Let us fix a set of generators $j_{\alpha,1}, \dots, j_{\alpha,k(\alpha)}$ in each Γ_α . Then

$$p_\alpha(g) := (\langle j_{\alpha,1}, g \rangle, \dots, \langle j_{\alpha,k(\alpha)}, g \rangle) \quad g \in G.$$

Now, for $\beta \geq \alpha$ and all i we have: $j_{\alpha,i} = \sum_{p=1}^{k(\beta)} c_{p,i} j_{\beta,p}$ for some $c_{p,i} \in \mathbb{Z}$ (the $c_{p,i}$'s may depend on α and β as well). In particular, the homomorphism p_α^β is defined by the formula:

$$p_\alpha^\beta(t_1, \dots, t_{k(\beta)}) := \left(\prod_{p=1}^{k(\beta)} t_p^{c_{p,1}}, \dots, \prod_{p=1}^{k(\beta)} t_p^{c_{p,k(\alpha)}} \right),$$

where $(t_1, \dots, t_{k(\beta)}) \in \mathbb{T}^{k(\beta)} := (\mathbb{T}^1)^{k(\beta)}$ (the direct product of $k(\beta)$ circles). Also, by $\bar{p} : \mathbb{T}^{k(0)} \rightarrow \mathbb{T}^3$ we denote similarly defined epimorphism such that $\tilde{p} = \bar{p} \circ p_0$.

Further, the set of pathwise connected components $C_n(G)$ of $GL(C(G)^{n \times n})$ is the direct limit of the family of pathwise connected components $C_n(\mathbb{T}^{k(\alpha)})$ of groups $GL(C(\mathbb{T}^{k(\alpha)})^{n \times n})$ equipped with pullback maps $(p_\alpha^\beta)^* : C_n(\mathbb{T}^{k(\beta)}) \rightarrow C_n(\mathbb{T}^{k(\alpha)})$ (indeed, we can approximate in the uniform topology any given element in a fixed pathwise connected component of $GL(C(G)^{n \times n})$ by elements X_m , $m = 1, 2, \dots$, of $P(G)$; in turn, each X_m (for sufficiently large m) belongs to a pathwise connected component of some $GL(C(\mathbb{T}^{\ell(m)})^{n \times n})$, where $\ell(m)$ is the number of generators of the subgroup of Γ generated by the finite set $\sigma(X_m)$). These sets have natural group structures induced by that of $GL(\mathbb{C}^{n \times n})$. The image of the set of pathwise connected components of $GLF(C(G)^{n \times n})$ in the set of pathwise connected components of $GL(C(G)^{n \times n})$ is a subgroup generated by pathwise connected components containing matrix functions $\text{diag}(1, \dots, 1, \langle j, \cdot \rangle)$, $j \in \Gamma$ (Theorem 5.2). In particular, this subgroup is an abelian subgroup of $C_n(G)$ (denoted further by $C_n(G_F)$).

Assuming, on the contrary, that $GLF(C(G)^{n \times n})$ is dense in $GL(C(G)^{n \times n})$ we obtain that $C_n(G_F) = C_n(G)$ and so this group is abelian. Let us consider the epimorphism $\tilde{p} : G \rightarrow \mathbb{T}^3$. Its pullback \tilde{p}^* induces a homomorphism $C_n(\mathbb{T}^3) \rightarrow C_n(G)$ of the corresponding groups of pathwise connected components. Under the above assumption we obtain that $\tilde{p}^*(C_n(\mathbb{T}^3)) \subset C_n(G_F)$. Let us take the connected component $[X] \in C_n(\mathbb{T}^3)$ containing the matrix $X := \text{diag}(1, \dots, 1, S \circ \psi_1)$, see Example 4.1. According to our assumption, $\tilde{p}^*([X]) \in C_n(G_F)$. In particular, $\tilde{p}^*(X)$ belongs to the pathwise connected component of $Y := \text{diag}(1, \dots, 1, \langle j, \cdot \rangle)$ for some $j \in \Gamma$. Next, let $\Gamma_+ \subset \Gamma$ be the minimal finitely generated subgroup containing $\tilde{\Gamma}$ and j . Our previous direct-inverse limit construction does not change if we start with $\Gamma_0 := \Gamma_+$. If $\mathbb{T}^{k(0)}$, $k(0) = 3$ or 4 , is the Pontryagin dual group to Γ_0 , and $p_0 : G \rightarrow \mathbb{T}^{k(0)}$ is the corresponding limiting map (in the inverse limit construction), then $\tilde{p}^*([X])$ and $[Y]$ are pullbacks by p_0 of connected components $[X']$ and $[Y']$ of $X' = (\bar{p}^*(X))$, and of some $Y' = \text{diag}(1, \dots, 1, \gamma)$ for $\gamma \in \Gamma_0$ from $C_n(\mathbb{T}^{k(0)})$. Since the set of pathwise connected components $C_n(G)$ of $GL(C(G)^{n \times n})$ is the direct limit of the family of pathwise connected components $C_n(\mathbb{T}^{k(\alpha)})$ of groups $GL(C(\mathbb{T}^{k(\alpha)})^{n \times n})$ equipped with pullback maps $(p_\alpha^\beta)^* : C_n(\mathbb{T}^{k(\beta)}) \rightarrow C_n(\mathbb{T}^{k(\alpha)})$, from the equality $[X] = [Y]$ it follows that there exists some Γ_α containing j such that $(\bar{p} \circ p_0^\alpha)^*(X) := (p_0^\alpha)^*(X')$ belongs to the pathwise connected component of

$$\text{diag}(1, \dots, 1, (p_\alpha^0)^*(\delta)) := (p_\alpha^0)^*(Y')$$

in $C_n(\mathbb{T}^{k(\alpha)})$.

Further, the kernel of the epimorphism $\bar{p} \circ p_0^\alpha : \mathbb{T}^{k(\alpha)} \rightarrow \mathbb{T}^3$ is a compact abelian group which is the direct product of a torus $H_1 \cong \mathbb{T}^{k(\alpha)-3}$ and a finite group H_2 . Thus the quotient group $\mathbb{T}^3(H_2) := \mathbb{T}^{k(\alpha)}/H_1$ is a finite covering of \mathbb{T}^3 with the covering group H_2 . By $r : \mathbb{T}^3(H_2) \rightarrow \mathbb{T}^3$ we denote the corresponding covering map; thus, the preimage of every point in \mathbb{T}^3 is identified with H_2 . We have

$$\mathbb{T}^{k(\alpha)} \rightarrow \mathbb{T}^{k(\alpha)}/H_1 = \mathbb{T}^3(H_2) \xrightarrow{r} \mathbb{T}^3.$$

We let also X_1 be the pullback $r^*(X)$ of X to $\mathbb{T}^3(H_2)$. One can easily see that X_1 does not belong to the pathwise connected component of $GL(C(\mathbb{T}^3(H_2))^{n \times n})$ containing 1. For otherwise, the map $S \circ \psi_1 \circ r : \mathbb{T}^3(H_2) \rightarrow SU_2$ is homotopic to a constant map $\mathbb{T}^3(H_2) \rightarrow SU_2$. Equivalently, $\psi_1 \circ r : \mathbb{T}^3(H_2) \rightarrow \mathbb{S}^3$ is homotopic to a constant map $\mathbb{T}^3(H_2) \rightarrow \mathbb{S}^3$. But the degree of r is a nonzero integer (because r is a finite covering of a torus) therefore from the latter statement and the Hopf theorem [19, Section II.8] it follows that $\psi_1 : \mathbb{T}^3 \rightarrow \mathbb{S}^3$ is homotopic to a constant map $\mathbb{T}^3 \rightarrow \mathbb{S}^3$. This contradicts our choice of ψ_1 , see Example 4.1.

Next, the quotient map $q : \mathbb{T}^{k(\alpha)} \rightarrow \mathbb{T}^3(H_2)$ admits a continuous right inverse i.e. $\mathbb{T}^{k(\alpha)}$ is isomorphic (as an abelian Lie group) to $H_1 \times \mathbb{T}^3(H_2)$. Indeed, since all fibres of q are connected, it determines a surjective homomorphism of fundamental groups $q_* : \pi_1(\mathbb{T}^{k(\alpha)}) \cong \mathbb{Z}^{k(\alpha)} \rightarrow \pi_1(\mathbb{T}^3(H_2)) \cong \mathbb{Z}^3$. In particular, there exists a monomorphism $m : \pi_1(\mathbb{T}^3(H_2)) \rightarrow \pi_1(\mathbb{T}^{k(\alpha)})$ such that $q_* \circ m = id$ and

$$\pi_1(\mathbb{T}^{k(\alpha)}) = \pi_1(H_1) \oplus m(\pi_1(\mathbb{T}^3(H_2))).$$

The map m is extended by linearity to the monomorphism of \mathbb{R} -linear hulls of the corresponding fundamental groups:

$$\widehat{m} : \langle \pi_1(\mathbb{T}^3(H_2)) \rangle_{\mathbb{R}} \cong \mathbb{R}^3 \rightarrow \langle \pi_1(\mathbb{T}^{k(\alpha)}) \rangle_{\mathbb{R}} \cong \mathbb{R}^{k(\alpha)},$$

equivariant (i.e., commuting) with respect to the actions of $\pi_1(\mathbb{T}^3(H_2))$ on $\langle \pi_1(\mathbb{T}^3(H_2)) \rangle_{\mathbb{R}}$ and of $\pi_1(\mathbb{T}^{k(\alpha)})$ on $\langle \pi_1(\mathbb{T}^{k(\alpha)}) \rangle_{\mathbb{R}}$ by translations. Therefore \widehat{m} determines a homomorphism of groups

$$\widetilde{m} : \mathbb{T}^3(H_2) = \langle \pi_1(\mathbb{T}^3(H_2)) \rangle_{\mathbb{R}} / \pi_1(\mathbb{T}^3(H_2)) \rightarrow \mathbb{T}^{k(\alpha)} = \langle \pi_1(\mathbb{T}^{k(\alpha)}) \rangle_{\mathbb{R}} / \pi_1(\mathbb{T}^{k(\alpha)}).$$

Similarly, $\mathbb{T}^{k(\alpha)} = \langle \pi_1(\mathbb{T}^{k(\alpha)}) \rangle_{\mathbb{R}} / \pi_1(\mathbb{T}^{k(\alpha)})$ and q is defined by the equivariant surjective map $\widehat{q}_* : \langle \pi_1(\mathbb{T}^{k(\alpha)}) \rangle_{\mathbb{R}} \rightarrow \langle \pi_1(\mathbb{T}^3(H_2)) \rangle_{\mathbb{R}}$ extending q_* by linearity. From here and the definition of q and m we obtain that $q \circ \widetilde{m} = id$.

Thus without loss of generality we may assume that X_1 is defined on the subgroup $\mathbb{T}^3(H_2)$ of $\mathbb{T}^{k(\alpha)}$. Under this assumption the pullback $(\bar{p} \circ p_0^\alpha)^*(X)$ of X_1 to $\mathbb{T}^{k(\alpha)}$ by means of q belongs to the pathwise connected component containing $\text{diag}(1, \dots, 1, (p_\alpha^0)^*(\delta))$. Hence, $(\bar{p} \circ p_0^\alpha)^*(X)$ and $\text{diag}(1, \dots, 1, (p_\alpha^0)^*(\delta))$ are homotopic maps of $\mathbb{T}^{k(\alpha)}$ into $GL(\mathbb{C}^{n \times n})$. The restriction of the homotopy between them to $\mathbb{T}^3(H_2)$ determines a homotopy between $\text{diag}(1, \dots, 1, (p_\alpha^0)^*(\delta)|_{\mathbb{T}^3(H_2)})$ and X_1 .

But X_1 maps $\mathbb{T}^3(H_2)$ into SU_2 and so its determinant is 1. The last two facts and Theorem 5.2 imply that $(p_\alpha^0)^*(\delta)|_{\mathbb{T}^3(H_2)}$ is homotopic to the unit homomorphism. In particular, X_1 belongs to the pathwise connected component of the group $GL(C(\mathbb{T}^3(H_2))^{n \times n})$ containing 1. This contradicts the choice of X .

Thus we have proved that $p_0^*(X)$ does not belong to the set $C_n(G_F)$ and therefore $C_n(G_F)$ is a proper subgroup of $C_n(G)$. This completes the proof of Theorem 5.1. \square

We indicate a couple of corollaries from the main results of this section:

Corollary 5.3 *Let Γ and \mathcal{B} be as in Theorem 2.1. Then, for every natural $n \geq 2$ there exist infinitely many pathwise connected components of $GL(\mathcal{B}^{n \times n})$ with the property that each one of these components does not contain any $C(G)$ -factorable element of $GL(\mathcal{B}^{n \times n})$.*

Proof. Follows immediately from Theorem 5.1 and Proposition 3.2. \square

Corollary 5.4 *Let Γ and \mathcal{B} be as in Theorem 2.1. Then there exists $A \in P(G)^{n \times n} \cap GL(\mathcal{B}^{n \times n})$ such that some neighborhood (in $\mathcal{B}^{n \times n}$) of A does not contain any $C(G)$ -factorable matrix function.*

Proof. Take A_0 in one of the pathwise connected components of $GL(\mathcal{B}^{n \times n})$ described in Corollary 5.3, and approximate A_0 by elements of $P(G)^{n \times n}$. \square

6 Subgroup generated by factorable matrix functions

We focus in the section on the properties of the subgroup generated by the factorable matrix functions. They will be needed for the proof of Theorem 2.1, but are also of independent interest.

An admissible algebra $\mathcal{B} \subseteq C(G)$ is said to be *decomposing* (with respect to the order \preceq) if $\mathcal{B} = \mathcal{B}_+ + \mathcal{B}_-$. This concept was introduced in [6], and studied extensively, see e.g. [9, 15]. Note that $W(G)$ is decomposing while $C(G)$ is not.

Lemma 6.1 *Assume \mathcal{B} is a decomposing admissible algebra. Then every element X of the pathwise connected component $GL_0(\mathcal{B}^{n \times n})$ of identity in $GL(\mathcal{B}^{n \times n})$, admits representation as a product of canonically \mathcal{B} -factorable matrix functions.*

Proof. Let Λ be an open neighborhood of identity in $GL_0(\mathcal{B}^{n \times n})$ such that every element of Λ admits a canonical \mathcal{B} -factorization (existence of such Λ follows e.g. from [15, Theorem XXIX.9.1]; this is where the decomposing property of \mathcal{B} is used). If $X \in GL_0(\mathcal{B}^{n \times n})$, then let $X(t)$, $0 \leq t \leq 1$, be a continuous path connecting X with I within $GL(\mathcal{B}^{n \times n})$; thus, $X(0) = X$, $X(1) = I$. Partition the interval $[0, 1]$, $0 = t_1 < t_2 < \dots < t_p < t_{p+1} = 1$, so that $X_{t_j} X_{t_{j+1}}^{-1} \in \Lambda$ for $j = 1, 2, \dots, p$. Then

$$X = \prod_{j=1}^p X_{t_j} X_{t_{j+1}}^{-1}$$

is a product of canonically \mathcal{B} -factorable matrices. \square

In the setting of decomposing admissible algebras \mathcal{B} , it is known that the set of canonically \mathcal{B} -factorable elements of $\mathcal{B}^{n \times n}$ is open (this is an easy corollary of [15, Theorem XXIX.9.1] which has been used in the proof of Lemma 6.1), and that in the classical case $G = \mathbb{T}$ and $\mathcal{B} = W(\mathbb{T})$, the set $GLF(W(\mathbb{T})^{n \times n})$ is open, as follows from [16]. In general, the set $GLF(W(G)^{n \times n})$ is not open, cf. the example at the end of Section 1.

Note also that for a decomposing admissible algebra \mathcal{B} , the group \mathcal{G} that consists of all finite products of elements of $GLF(\mathcal{B}^{n \times n})$ and their inverses, is open and closed in $GL(\mathcal{B}^{n \times n})$. Indeed, assume $G \in GL(\mathcal{B}^{n \times n})$ is in the closure of \mathcal{G} ; then for every $\epsilon > 0$ there is $G_\epsilon \in \mathcal{G}$ such that $\|G - G_\epsilon\| < \epsilon$ (the norms here are taken with respect to \mathcal{B}). Letting $H_\epsilon := G^{-1}G_\epsilon$ we see that $\|H_\epsilon - I\| \leq \|G^{-1}\|\epsilon$, and by taking ϵ sufficiently small we guarantee that H_ϵ admits a canonical \mathcal{B} -factorization. Now clearly $G = G_\epsilon H_\epsilon^{-1}$ belongs to \mathcal{G} . The openness of \mathcal{G} can be proved by similar arguments. It follows that \mathcal{G} coincides with the minimal closed subgroup of $GL(\mathcal{B}^{n \times n})$ that contains $GLF(\mathcal{B}^{n \times n})$ (again, for a decomposing admissible algebra \mathcal{B}).

For an admissible algebra \mathcal{B} , we let $\mathcal{D}(\mathcal{B}^{n \times n})$ be the union of all those pathwise connected components of $GL(\mathcal{B}^{n \times n})$ that contain an element of the form $\text{diag}(1, \dots, 1, \langle j, \cdot \rangle)$, $j \in \Gamma$. Clearly, $\mathcal{D}(\mathcal{B}^{n \times n})$ is a closed and open subgroup of $GL(\mathcal{B}^{n \times n})$.

Theorem 6.2 *Let \mathcal{B} be an admissible algebra. Then the minimal closed subgroup of $GL(\mathcal{B}^{n \times n})$ containing $GLF(\mathcal{B}^{n \times n})$ is a subgroup of $\mathcal{D}(\mathcal{B}^{n \times n})$.*

If in addition \mathcal{B} is decomposing, then the minimal closed subgroup of $GL(\mathcal{B}^{n \times n})$ containing $GLF(\mathcal{B}^{n \times n})$ coincides with $\mathcal{D}(\mathcal{B}^{n \times n})$.

Proof. For the first part of the theorem it suffices to prove that every element $A \in GLF(\mathcal{B}^{n \times n})$ can be connected within $GLF(\mathcal{B}^{n \times n})$ to an element of the form $\text{diag}(1, \dots, 1, \langle j, \cdot \rangle)$, $j \in \Gamma$ by a continuous path (with respect to the \mathcal{B} -norm).

Let

$$A(g) = A_-(g) (\text{diag}(\langle j_1, g \rangle, \dots, \langle j_n, g \rangle)) A_+(g), \quad g \in G, \quad A_\pm, A_\pm^{-1} \in \mathcal{B}_\pm^{n \times n}, \quad (6.1)$$

be a \mathcal{B} -factorization of A . A known argument (see [5, Lemma 6.4] and the proof of [5, Theorem 6.3]) shows that $\text{diag}(\langle j_1, \cdot \rangle, \dots, \langle j_n, \cdot \rangle)$ can be connected to $\text{diag}(1, \dots, 1, \langle j_1 + \dots + j_n, \cdot \rangle)$ within $GLF(\mathcal{B}^{n \times n})$ by a continuous path. Thus, we need only to show that A_\pm can be connected to the constant I within $GL(\mathcal{B}_\pm^{n \times n})$ by a continuous path. But this follows from Proposition 3.2, part (2), and the proof of the first part of the theorem is completed.

For the second part, suppose $X \in \mathcal{D}(\mathcal{B}^{n \times n})$. Then $X = X_1 X_0$, where $X_1 = \text{diag}(1, \dots, 1, \langle j, \cdot \rangle)$ for some $j \in \Gamma$ and $X_0 \in GL_0(\mathcal{B}^{n \times n})$. Now X_1 is obviously \mathcal{B} -factorable, and X_0 is a product of canonically \mathcal{B} -factorable matrices by Lemma 6.1. Thus, $\mathcal{D}(\mathcal{B}^{n \times n})$ is contained in the minimal closed subgroup of $GL(\mathcal{B}^{n \times n})$ containing all \mathcal{B} -factorable matrices. \square

Notice that Lemma 6.1 and Theorem 6.2 are valid for any connected compact abelian group G , not only for those whose dual group Γ contains \mathbb{Z}^3 .

The case of the two-dimensional torus is perhaps interesting:

Corollary 6.3 *Let $G = \mathbb{T}^2$, and let \mathcal{B} be a decomposing admissible algebra. Then $GL(\mathcal{B}^{n \times n})$ is equal to the minimal closed subgroup of $GL(\mathcal{B}^{n \times n})$ containing $GLF(\mathcal{B}^{n \times n})$.*

Proof. By Theorem 6.2, the minimal closed subgroup of $GL(\mathcal{B}^{n \times n})$ containing $GLF(\mathcal{B}^{n \times n})$ coincides with $\mathcal{D}(\mathcal{B}^{n \times n})$. Using Proposition 3.2 and the proof of Proposition 3.4 we see that in turn $\mathcal{D}(\mathcal{B}^{n \times n})$ coincides with $GL(\mathcal{B}^{n \times n})$. Indeed, the pathwise connected components of both groups are parametrized identically by \mathbb{Z}^2 . \square

Proof of Theorem 2.1. As we have seen in the proof of Theorem 5.1 (see also Proposition 3.2), there are infinitely many pathwise connected components of $GL(\mathcal{B}^{n \times n})$ that do not intersect $\mathcal{D}(\mathcal{B}^{n \times n})$. The result now follows from Theorem 6.2. \square

7 Scalar valued functions

In this section we consider the scalar case.

Theorem 7.1 *Let $\mathcal{B} \subseteq C(G)$ be an admissible algebra, where G is a connected compact abelian group. Then:*

- (a) *The set $GLF(\mathcal{B})$ of \mathcal{B} -factorable scalar functions is dense in $GL(\mathcal{B})$;*
- (b) *The equality $GLF(\mathcal{B}) = GL(\mathcal{B})$ holds if and only if \mathcal{B} is decomposing.*

In the classical case $G = \mathbb{T}$ part (b) is well known, see e.g. [17, Theorem 3.1].

For the proof of Theorem 7.1 it will be convenient to start with preliminary results. The next proposition is perhaps independently interesting and holds for matrix functions as well.

Proposition 7.2 *Let \mathcal{B} be an admissible algebra, $f \in \mathcal{B}^{n \times n}$, and let*

$$\Omega = \{z \in \mathbb{C} : z \text{ is an eigenvalue of } f(g) \text{ for some } g \in G\}.$$

If Ψ is an analytic function in an open neighborhood of the closure of Ω , then $\Psi \circ f \in \mathcal{B}^{n \times n}$.

Here, for every fixed $g \in G$, $\Psi \circ f(g)$ is understood as the $n \times n$ matrix defined by the standard functional calculus.

The proof is essentially the same as that of [30, Proposition 2.3], [29, Proposition 2.3], where it was proved for $W(G)$, and with Γ an additive subgroup of \mathbb{R}^k (with the discrete topology).

Proposition 7.3 *A character $c \in \Gamma$ is an exponential, i.e. $\langle c, \cdot \rangle = e^{f(\cdot)}$ for some $c \in \Gamma$ and $f \in C(G)$, if and only if $c = 0$.*

Proof. For the reader's convenience we supply a known proof. The “if” part is obvious. For the “only if” part, we have

$$e^{f(gh)} = \langle c, gh \rangle = \langle c, g \rangle \cdot \langle c, h \rangle = e^{f(g)+f(h)}, \quad \forall g, h \in G.$$

Therefore

$$f(gh) - f(g) - f(h) = 2\pi i a(g, h), \quad (7.1)$$

where $a(g, h)$ is an integer valued continuous function on $G \times G$. Since G is connected, $G \times G$ is also connected and therefore a is a constant. Now if $g = h = 1$, then (7.1) gives

$$a(g, h) = a(1, 1) = \frac{-f(1)}{2\pi i}.$$

For a fixed $g \in G$ by applying inductively (7.1) we have for any natural n :

$$f(g^n) = nf(g) - (n-1)f(1) = f(g) + (n-1)(f(g) - f(1)). \quad (7.2)$$

Since G is compact, the image $f(G)$ is a bounded subset of the plane. On the other hand (7.2) shows that this is possible for $n \rightarrow \infty$ if and only if $f(g) = f(1)$ for any $g \in G$. Thus f is constant and moreover $f \in 2\pi i\mathbb{Z}$. From here we get $\langle c, \cdot \rangle = e^{f(1)} = 1$. \square

Proof of Theorem 7.1. Part (a). We use approximation (in the norm of \mathcal{B}) of any given element of $GL(\mathcal{B})$ by elements of $P(G) \cap GL(\mathcal{B})$, thereby reducing the proof to the case of finitely generated subgroups Γ' of Γ rather than Γ itself.

Indeed, letting $C(G)_{\Gamma'}$ stand for the closure in $C(G)$ of the set $P(G)_{\Gamma'}$ of elements of $P(G)$ with the Bohr-Fourier spectra in Γ' , we first note that $C(G)_{\Gamma'}$ coincides with the set of elements of $C(G)$ having the Bohr-Fourier spectra in Γ' (see e.g. [10, Theorem 7.14]). Next, let G' be the dual group of Γ' , and observe the identification $C(G') \cong C(G)_{\Gamma'}$. This follows easily from the fact that Γ' is the dual group of $G' \cong G/H$, where $H := \{g \in G: \langle \gamma, g \rangle = 1 \ \forall \ \gamma \in \Gamma'\}$ is the annihilator of Γ' and a closed subgroup of G [32, Section 2.1]; thus, if $a \in P(G)_{\Gamma'}$, then a is constant on every coset of G by H , and therefore every element of $C(G)_{\Gamma'}$ is also constant on every coset of G by H . Also, using the approximation of any given element of $GL(\mathcal{B})$ by elements of $P(G) \cap GL(\mathcal{B})$, we replace \mathcal{B} with the closure (in the norm of \mathcal{B}) of $P(G)_{\Gamma'}$; denote this closure by \mathcal{B}' . Note that \mathcal{B}' can be understood also as a subalgebra of $C(\Gamma')$ via the above identification $C(G') \cong C(G)_{\Gamma'}$; the inverse closedness of \mathcal{B}' follows by identifying the elements of \mathcal{B}' with those functions in \mathcal{B} that are constant on every coset of G by G' . The finitely generated subgroups Γ' are further identified with \mathbb{Z}^q , for suitable integers q .

Next, we prove that every element $a \in P(\mathbb{T}^q) \cap GL(\mathcal{B})$ admits a decomposition of the form

$$a = |a| \langle c, \cdot \rangle e^u = e^b \langle c, \cdot \rangle e^u, \quad (7.3)$$

for some $c \in \mathbb{Z}^q$ and $u, b \in \mathcal{B}$. The proof of (7.3) will mimic that of [31, Proposition 2.4], [30, Theorem 2.4], where (7.3) was proved for the case $\mathcal{B} = W(\mathbb{T}^q)$. First note that $a\bar{a} \in \mathcal{B}$, hence by Proposition 7.2, $|a| = \sqrt{a\bar{a}}$ and $\log |a|$ belong to \mathcal{B} . Thus, the second equality in (7.3) follows. For the first equality, note that it was proved in [27] with $u \in C(\mathbb{T}^q)$. Thus, we need only to show that in fact $u \in \mathcal{B}$. Represent the function u as the sum $u = u_0 + u_1$, where $u_0 \in \mathcal{B}$ and $\|u_1\|_{C(\mathbb{T}^q)} < \frac{\pi}{2}$. Then

$$e^{u_1(g)} = a(g)|a(g)|^{-1}\langle -c, g \rangle e^{-u_0(g)}, \quad g \in \mathbb{T}^q. \quad (7.4)$$

By Proposition 7.2 the function $|a|^{-1}$ belongs to \mathcal{B} . The other two multiples in the right hand side of (7.4) obviously belong to \mathcal{B} ; hence, so does the function $\xi = e^{u_1}$. On the other hand, the values of ξ lie in the right open half-plane. Using Proposition 7.2 again, we may define $u_2 \in \mathcal{B}$ so that $e^{u_2} = \xi$. Since u_1 and u_2 are both continuous on \mathbb{T}^q , this means that they differ by a constant summand. But then u_1 belongs to \mathcal{B} simultaneously with u_2 . Finally, the function $u = u_0 + u_1$ belongs to \mathcal{B} as well.

Now, for $a \in P(\mathbb{T}^q) \cap GL(\mathcal{B})$ and its decomposition (7.3), let $\{u_m\}_{m=1}^\infty, \{b_m\}_{m=1}^\infty$ be sequences such that $u_m, b_m \in \mathcal{B}_+ + \mathcal{B}_-$ (where $\mathcal{B}_+ + \mathcal{B}_-$ is considered with its Banach algebra norm) and

$$\lim_{m \rightarrow \infty} u_m = u, \quad \lim_{m \rightarrow \infty} b_m = b$$

in the \mathcal{B} -norm. Then clearly

$$a_m := e^{b_m} \langle c, \cdot \rangle e^{u_m} \longrightarrow a \quad (7.5)$$

in the \mathcal{B} -norm as $m \rightarrow \infty$. On the other hand, it follows from (7.5) that every a_m admits a \mathcal{B} -factorization. Indeed, write $u_m = u_{m+} + u_{m-}$, $b_m = b_{m+} + b_{m-}$ with $u_{m\pm}, b_{m\pm} \in \mathcal{B}_\pm$. Then

$$a_m = e^{b_{m-}} e^{u_{m-}} \langle c, \cdot \rangle e^{b_{m+}} e^{u_{m+}}$$

is the desired factorization. This proves the density of $GLF(\mathcal{B})$ in $GL(\mathcal{B})$.

Part (b), the “if” part. Assume \mathcal{B} is decomposing, and let $a \in GL(\mathcal{B})$. Approximate a in the \mathcal{B} -norm by a function $b \in P(G) \cap GL(\mathcal{B})$ for which the function ab^{-1} is so close to identity that it is canonically \mathcal{B} -factorable (see e.g. [15, Theorem XXIX.9.1]; the decomposing property of \mathcal{B} is essential here). As proved in the preceding paragraph, we can approximate b by $c \in GL(\mathcal{B})$, again in the \mathcal{B} -norm, so that c admits a \mathcal{B} -factorization and bc^{-1} is canonically \mathcal{B} -factorable (we use [15, Theorem XXIX.9.1] again). Now $a = c \cdot (bc^{-1}) \cdot (ab^{-1})$, and since every factor admits a \mathcal{B} -factorization, the \mathcal{B} -factorability of a follows.

Part (b), the “only if” part. Assume that every invertible element of \mathcal{B} is \mathcal{B} -factorable. Let $h \in \mathcal{B}$ and let $f = e^h$. Since $f \in GL(\mathcal{B})$, we have a \mathcal{B} -factorization $f = b_- \langle c, \cdot \rangle b_+$, $c \in \Gamma$, $b_\pm \in GL(\mathcal{B}_\pm)$. By Proposition 3.2, part (2), b_\pm belongs to the pathwise connected component of identity in $GL(\mathcal{B}_\pm)$. Thus, the character $\langle c, \cdot \rangle$

belongs to the same pathwise connected component of $GL(\mathcal{B})$ as e^h does. But this pathwise connected component is just the component that contains the constant 1, or equivalently, the set of all exponentials e^s , $s \in \mathcal{B}$. Thus, $\langle c, \cdot \rangle$ is an exponential, and by Proposition 7.3 we obtain $c = 0$. Furthermore, since b_\pm belongs to the pathwise connected component of identity in $GL(\mathcal{B}_\pm)$, we have $b_\pm = e^{d_\pm}$ for some $d_\pm \in \mathcal{B}_\pm$. Now $e^h = e^{d_-} e^{d_+}$, and so $h = d_- + d_+ + 2\pi i k$, where $k : G \rightarrow \mathbb{Z}$ is a continuous function. So k must be constant, and we are done. \square

8 Non-denseness of triangularizable matrix functions

In this section we apply our main result to the problem of triangularizability. As everywhere in the paper, let G be a compact abelian group with the dual group Γ . Let \mathcal{B} be an admissible algebra. An element $A \in \mathcal{B}^{n \times n}$ is said to be (right) \mathcal{B} -triangularizable if A admits a representation (1.4), where the diagonal term $\text{diag}(\langle j_1, g \rangle, \dots, \langle j_n, g \rangle)$ is replaced by a triangular matrix $T = [t_{ij}]_{i,j=1}^n$, with $t_{ij} \in \mathcal{B}$ for $i, j = 1, \dots, n$, $t_{ij} = 0$ if $i > j$, and the diagonal elements t_{11}, \dots, t_{nn} belong to $GL(\mathcal{B})$. Denote by $GLT(\mathcal{B}^{n \times n})$ the set of $n \times n$ \mathcal{B} -triangularizable matrix functions. Clearly,

$$GLF(\mathcal{B}^{n \times n}) \subseteq GLT(\mathcal{B}^{n \times n}) \subseteq GL(\mathcal{B}^{n \times n}).$$

An example ([21], see also [4, Theorem 8.17]) shows that, for $\Gamma = \mathbb{R}$, there exist 2×2 triangular matrix functions in $P(G) \cap GL(W(G)^{2 \times 2})$ that are not $W(G)$ -factorable.

The following question has been proposed, in the context of the algebra $W(G)$: *Does $GL(\mathcal{B}^{n \times n}) = GLT(\mathcal{B}^{n \times n})$ hold for admissible algebras?* Theorem 2.1 implies that generally the answer is no. More precisely, we have the following result. Denote by $\mathcal{T}(\mathcal{B}^{n \times n})$ the minimal closed subgroup of $GL(\mathcal{B}^{n \times n})$ that contains $GLT(\mathcal{B}^{n \times n})$.

Theorem 8.1 *Let Γ be a torsion free abelian group that contains a subgroup isomorphic to \mathbb{Z}^3 , and let \mathcal{B} be an admissible algebra. Then, for every natural $n \geq 2$ there exist infinitely many pathwise connected components of $GL(\mathcal{B}^{n \times n})$ with the property that each one of these components does not intersect $\mathcal{T}(\mathcal{B}^{n \times n})$. In particular, $\mathcal{T}(\mathcal{B}^{n \times n})$ is not dense in $GL(\mathcal{B}^{n \times n})$.*

Proof. Let $A \in GLT(\mathcal{B}^{n \times n})$, so

$$A(g) = A_-(g) ([t_{ij}(g)]_{i,j=1}^n) A_+(g), \quad g \in G,$$

where $A_\pm, A_\pm^{-1} \in \mathcal{B}_\pm^{n \times n}$, $t_{ij} \in \mathcal{B}$, $t_{ij} = 0$ if $i > j$, and $t_{jj} \in GL(\mathcal{B})$. By the pathwise connectivity of $GL(\mathcal{B}_\pm^{n \times n})$ (Proposition 3.2 part (2)), we see that there is a continuous path in $GL(\mathcal{B}^{n \times n})$ connecting A with $[t_{ij}]_{i,j=1}^n$. In turn, $[t_{ij}]_{i,j=1}^n$ is obviously continuously connected in $GL(\mathcal{B}^{n \times n})$ with $\text{diag}(t_{11}, \dots, t_{nn})$. By Theorem 7.1, each t_{jj} can be

approximated arbitrarily well (in the \mathcal{B} -norm) by \mathcal{B} -factorable scalar functions, hence $\text{diag}(t_{11}, \dots, t_{nn})$ can be approximated arbitrarily well by \mathcal{B} -factorable matrix functions. Thus, A belongs to a pathwise connected component of $GL(\mathcal{B}^{n \times n})$ that contains a \mathcal{B} -factorable matrix function. By Theorem 6.2, the union of those pathwise connected components of $GL(\mathcal{B}^{n \times n})$ is contained in $\mathcal{D}(\mathcal{B}^{n \times n})$, and since $\mathcal{D}(\mathcal{B}^{n \times n})$ is an open and closed subgroup of $GL(\mathcal{B}^{n \times n})$, also $\mathcal{T}(\mathcal{B}^{n \times n})$ is contained in $\mathcal{D}(\mathcal{B}^{n \times n})$. It remains to observe that there are infinitely many pathwise connected components of $GL(\mathcal{B}^{n \times n})$ that do not intersect $\mathcal{D}(\mathcal{B}^{n \times n})$ (cf. the proof of Theorem 2.1). \square

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